

BORN–INFELD EQUATIONS*

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Classical linear vacuum electrodynamics with point massive charged particles has two limiting properties: the electromagnetic energy of a point particle field is infinity, and a Lorentz force must be postulated to describe interactions between point particles and an electromagnetic field. Nonlinear vacuum electrodynamics can be free of these imperfections.

Gustav Mie (1912–13) considered a nonlinear electrodynamics model in the framework of his “Fundamental unified theory of matter”. In this theory the electron is represented by a nonsingular solution with a finite electromagnetic energy, but Mie’s field equations are non-invariant under the gauge transformation for an electromagnetic four-potential (addition of the four-gradient of an arbitrary scalar function).

Max Born (1934) considered a nonlinear electrodynamics model that is invariant under the gauge transformation. A stationary electron in this model is represented by an electrostatic field configuration that is everywhere finite, in contrast to the case of linear electrodynamics when the electron’s field is infinite at the singular point (see Figure 1). The central point in Born’s electron is also singular because there is a discontinuity of electrical displacement field at this point (hedgehog singularity). The full electromagnetic energy of this electron’s field configuration is finite.

Born and Leopold Infeld (1934) then considered a more general nonlinear electrodynamics model which has the same solution associated with electron. Called Born–Infeld electrodynamics, this model is based on Born–Infeld equations, which have the form of Maxwell’s equations, including electrical and magnetic field strengths \mathbf{E} , \mathbf{H} , and inductions \mathbf{D} , \mathbf{B}

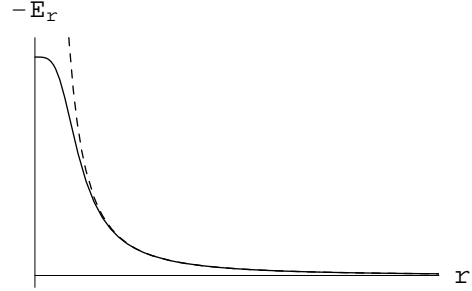


Figure 1: Radial components of electrical field for Born’s electron and purely Coulomb field (dashed).

with nonlinear constitutive relations $\mathbf{D} = \mathbf{D}(\mathbf{E}, \mathbf{B})$, $\mathbf{H} = \mathbf{H}(\mathbf{E}, \mathbf{B})$ of a special kind. For inertial reference frames and in the region outside of field singularities, these equations are

$$\begin{cases} \operatorname{div} \mathbf{B} = 0, \\ \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} + \operatorname{curl} \mathbf{E} = 0, \\ \operatorname{div} \mathbf{D} = 0, \\ \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} - \operatorname{curl} \mathbf{H} = 0, \end{cases} \quad (1)$$

where

$$\begin{aligned} \mathbf{D} &= \frac{1}{L} (\mathbf{E} + \chi^2 \mathcal{J} \mathbf{B}), \\ \mathbf{H} &= \frac{1}{L} (\mathbf{B} - \chi^2 \mathcal{J} \mathbf{E}), \end{aligned} \quad (2)$$

$$\begin{aligned} L &\equiv \sqrt{|1 - \chi^2 \mathcal{I} - \chi^4 \mathcal{J}^2|}, \\ \mathcal{I} &= \mathbf{E} \cdot \mathbf{E} - \mathbf{B} \cdot \mathbf{B}, \quad \mathcal{J} = \mathbf{E} \cdot \mathbf{B}. \end{aligned} \quad (3)$$

*Entry in Encyclopedia of Nonlinear Science, ed. Alwyn Scott. New York and London: Routledge, 2004, pp. 67–69; <http://www.routledge-ny.com/ref/nonlinearsci/>.

Relations (2) can be resolved for \mathbf{E} and \mathbf{H} :

$$\begin{aligned}\mathbf{E} &= \frac{1}{\mathcal{H}} (\mathbf{D} - \chi^2 \mathcal{P} \times \mathbf{B}) , \\ \mathbf{H} &= \frac{1}{\mathcal{H}} (\mathbf{B} + \chi^2 \mathcal{P} \times \mathbf{D}) ,\end{aligned}\quad (4)$$

where $\mathcal{H} = \sqrt{1 + \chi^2 (\mathbf{D}^2 + \mathbf{B}^2) + \chi^4 \mathcal{P}^2}$, $\mathcal{P} \equiv \mathbf{D} \times \mathbf{B}$. Using relations (4) for equations (1), the fields \mathbf{D} and \mathbf{B} are unknown.

The symmetrical energy-momentum tensor for Born-Infeld equations have the following components:

$$\begin{aligned}T^{00} &= \frac{1}{4\pi\chi^2} (\mathcal{H} - 1) , \quad T^{0i} = \frac{1}{4\pi} \mathcal{P}^i , \\ T^{ij} &= \frac{1}{4\pi} \{ \delta^{ij} [\mathbf{D} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{H} - \chi^{-2} (\mathcal{H} - 1)] \\ &\quad - (D^i E^j + B^i H^j) \} .\end{aligned}\quad (5)$$

In spherical coordinates the field of static Born's electron solution may have only radial components

$$D_r = \frac{e}{r^2} , \quad E_r = \frac{e}{\sqrt{\bar{r}^4 + r^4}} , \quad (6)$$

where e is the electron's charge and $\bar{r} \equiv \sqrt{|\chi e|}$. At the point $r = 0$ the electrical field has the maximum absolute value

$$|E_r(0)| = \frac{|e|}{\bar{r}^2} = \frac{1}{\chi} , \quad (7)$$

which Born and Infeld called the absolute field constant.

The energy of field configuration (6) is

$$m = \int T^{00} dV = \frac{2}{3} \beta \frac{\bar{r}^3}{\chi^2} , \quad (8)$$

where the volume integral is calculated over the whole space, and

$$\beta \equiv \int_0^\infty \frac{dr}{\sqrt{1 + r^4}} = \frac{[\Gamma(\frac{1}{4})]^2}{4\sqrt{\pi}} \approx 1.8541 . \quad (9)$$

In view of the definition for \bar{r} below (6), Equation (8) yields

$$\bar{r} = \frac{2}{3} \beta \frac{e^2}{m} . \quad (10)$$

Considering m as the mass of electron and using (7), Born & Infeld (1934) estimated the absolute field constant $\chi^{-1} \approx 3 \cdot 10^{20}$ V/m. Later Born & Schrödinger (1935) gave a new estimate (two orders of magnitude less) based on some considerations taking into account the spin of electron. (Of course, such estimates may be corrected with more detailed models.)

An electrically charged solution of the Born-Infeld equations can be generalized to a solution with the singularity having both electrical and magnetic charges (Chernitskii, 1999). A corresponding hypothetical particle is called a dyon (Schwinger, 1969). Nonzero (radial) components of fields for this solution have the form

$$\begin{aligned}D_r &= \frac{\mathcal{E}}{r^2} , \quad E_r = \frac{\mathcal{E}}{\sqrt{\bar{r}^4 + r^4}} , \\ B_r &= \frac{\mathcal{G}}{r^2} , \quad H_r = \frac{\mathcal{G}}{\sqrt{\bar{r}^4 + r^4}} ,\end{aligned}\quad (11)$$

where \mathcal{E} is electric charge and \mathcal{G} is magnetic one, $\bar{r} \equiv [\chi^2 (\mathcal{E}^2 + \mathcal{G}^2)]^{1/4}$. The energy of this solution is given by formula (8) with this definition for \bar{r} . It should be noted that space components of electromagnetic potential for the static dyon solution have a line singularity.

A generalized Lorentz force appears when a small almost constant field $\tilde{\mathbf{D}}$, $\tilde{\mathbf{B}}$ is considered in addition to the moving dyon solution. The sum of the field $\tilde{\mathbf{D}}$, $\tilde{\mathbf{B}}$ and the field of the dyon with varying velocity is taken as an initial approximation to some exact solution. Conservation of total momentum gives the following trajectory equation (Chernitskii, 1999):

$$\begin{aligned}m \frac{d}{dt} \frac{\mathbf{v}}{\sqrt{1 - \mathbf{v}^2}} &= \mathcal{E} (\tilde{\mathbf{D}} + \mathbf{v} \times \tilde{\mathbf{B}}) \\ &\quad + \mathcal{G} (\tilde{\mathbf{B}} - \mathbf{v} \times \tilde{\mathbf{D}}) ,\end{aligned}\quad (12)$$

where \mathbf{v} is the velocity of the dyon, m is the energy for static dyon defined by (8).

A solution with two dyon singularities (called a bidyon) having equal electric ($\mathcal{E} = e/2$) and opposite magnetic charges can be considered as a model for a charged particle with spin (Chernitskii, 1999). Such

a solution has both angular momentum and magnetic moment.

A plane electromagnetic wave with arbitrary polarization and form in the direction of propagation (without coordinate dependence in a perpendicular plane) is an exact solution to Born–Infeld equations. The simplest case assumes one nonzero component of the vector potential ($A_y \equiv \phi(t, x)$), whereupon Equations (1) reduce to the linearly polarized plane wave equation

$$(1 + \chi^2 \phi_x^2) \phi_{tt} - \chi^2 2 \phi_x \phi_t \phi_{xt} - (c^2 - \chi^2 \phi_t^2) \phi_{xx} = 0 \quad (13)$$

with indices indicating partial derivatives. Sometimes called the Born–Infeld equation, Equation (13) has solutions $\phi = \zeta(x^1 - x^0)$ and $\phi = \zeta(x^1 + x^0)$, where $\zeta(x)$ is arbitrary function (Whitham, 1974). Solutions comprising two interacting waves propagating in opposite directions is obtained via a hodograph transform (Whitham, 1974). Brunelli & Ashok (1998) have found a Lax representation for solutions of this equation.

A solution to the Born–Infeld equations which is the sum of two circularly polarized waves propagating in different directions was obtained by Erwin Schrödinger (1943).

Equations (1) with relations (2) have an interesting characteristic equation (Chernitskii, 1998):

$$\mathbf{g}^{\mu\nu} \frac{\partial \Phi}{\partial x^\mu} \frac{\partial \Phi}{\partial x^\nu} = 0, \quad \mathbf{g}^{\mu\nu} \equiv g^{\mu\nu} - 4\pi \chi^2 T^{\mu\nu}, \quad (14)$$

where $\Phi(x^\mu) = 0$ is an equation of the characteristic surface and $T^{\mu\nu}$ are defined by (5). This form for $\mathbf{g}^{\mu\nu}$, including additively the energy-momentum tensor, is special for Born–Infeld equations.

The Born–Infeld model appears also in quantized string theory (Fradkin & Tseytlin, 1985) and in Einstein’s unified field theory with a nonsymmetrical metric (Chernikov & Shavokhina, 1986). In general, this nonlinear electrodynamics model is connected with ideas of space-time geometrization and general relativity (see Eddington, 1924; Chernitskii, 2002).

*See also*¹ **Einstein equations; Hodograph trans-**

¹in Encyclopedia of Nonlinear Science, cited above.

form; Matter, nonlinear theory of; String theory

Further Reading

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